ICASE

FINITE ELEMENT METHODS OF THE LEAST SQUARES

TYPE FOR REGIONS WITH CORNERS

George J. Fix

Ernst Stephan

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING

NASA Langley Research Center, Hampton, Virginia 23665

Operated by the

UNIVERSITIES SPACE USRA RESEARCH ASSOCIATION

FINITE ELEMENT METHODS OF THE LEAST SQUARES

TYPE FOR REGIONS WITH CORNERS

G. J. Fix
Carnegie-Mellon University

E. Stephan
Carnegie-Mellon University
and
Technische Hochschule-Darmstadt, Federal Republic of Germany

ABSTRACT

This paper treats problems with corner singularities. It is shown that if appropriate weights are used in the least squares formulation, then optimal error estimates can be derived in unweighted L_2 norms.

1. Introduction

Least squares methods have been useful for approximating solutions to elliptic systems which are not strictly coercive. The Helmholtz equation is perhaps the most important example ([1] - [2]). The main advantage of the least squares formulation in this context, as compared with a standard Galerkin formulation, is the fact that it always produces a Hermitian positive definite system of algebraic equations. This is particularly attractive for three dimensional problems where storage considerations make iterative methods like S.O.R. desirable. The Stein-Ostrowski Theorem [3] states that for Hermitian systems this iterative method converges if and only if the system is positive definite.

The least squares formulation does however have one glaring defect, namely the extreme regularity on the solution that is needed to obtain convergence and stability. For example, a common least squares approximation to

$$(1.1) \qquad\qquad \Delta\phi + q\phi = f \quad \text{in} \quad \Omega$$

$$(1.2) \qquad\qquad \phi = 0 \quad \text{on} \quad \partial\Omega$$

is to require that

$$(1.3) \qquad\qquad \int_\Omega \{ |\text{grad } \phi - \underline{u}|^2 + |\text{div } \underline{u} + q\phi - f|^2 \}$$

be minimized as ϕ and $\underline{u}$ vary over appropriate finite dimensional spaces. The usual choice is let $\phi, \underline{u}$ be piecewise polynomials of degree $k - 1$ on a grid with generic mesh spacing h. These spaces have the property that smooth functions can be approximated to $O(h^k)$ in L_2. However, for this to be true of the least square approximations one needs, among other things, the

following regularity property [1]. There is a number $0 < C < \infty$ such that for any function f in the Sobolev space $H^1(\Omega)$ there is a unique solution ϕ of (1.1) − (1.2) such that

(1.4)
$$\|\phi\|_{H^3(\Omega)} \leq C\|f\|_{H^1(\Omega)} .$$

This result is valid for only smooth regions Ω, and in particular it is not valid if Ω has corners. Moreover, numerical experiments indicate that something like (1.4) may actually be necessary. For example, a series of numerical experiments [4] with regions with cracks have shown that this approach produces substandard results even with a rather extreme mesh refinement near the corner.

In this paper we consider an alternate least squares approximation in weighted spaces. These are spaces where the analogs of (1.4) are valid if the appropriate weights are used. The most important aspect of our error analysis is that estimates can be derived in unweighted spaces for the approximations. Numerical experiments indicate that the use of these weighted spaces give a significant increase in accuracy, although refinement near the crack is still needed as is to be expected.

For simplicity we shall consider planar regions Ω with only one corner having interior angle θ_0 as shown in Figure 1. Our results are restricted to the case $0 \leq \theta_0 < 2\pi$ because of the crucial role played in the analysis by the Hardy-Littlewood inequality and continuous embedding in weighted Sobolev spaces. The latter is known to be valid only when Ω has the cone property, i.e., $\theta \neq 2\pi$. Moreover, the regularity results used for the weighted Sobolev spaces have only been developed for planar or conical regions in [6], hence our restriction to the planar case. Neumann and mixed problems could be treated analogously. For brevity we consider here only the Dirichlet problem.
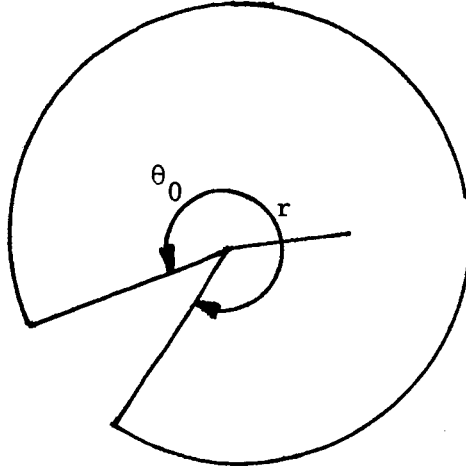
FIGURE 1

The planar region Ω

2. Formulation of the Approximation

For the number $\alpha > 0$ define

(2.1)
$$\|\underline{v}\|_{1,\alpha}^2 = \|r^{\alpha/2} \operatorname{div} \underline{v}\|_{L^2(\Omega)}^2 + \|\underline{v}\|_{[L^2(\Omega)]^2}^2 ,$$

and let $\underline{W}_\alpha^1(\Omega)$ be the closure of $[C^\infty(\overline{\Omega})]^2$ in this norm. Let

(2.2)
$$S_\delta \subseteq \overset{o}{H}{}^1(\Omega) \qquad \underline{V}_h \subseteq \underline{W}_\alpha^1(\Omega)$$

be finite dimensional spaces. We seek

(2.3)
$$\phi_\delta \in S_\delta, \qquad \underline{u}_h \in \underline{V}_h$$

which minimize

$$(2.4) \qquad \int_\Omega \{ \, |\text{grad } \psi^\delta - \underline{v}^h|^2 + r^\alpha |\text{div } \underline{v}^h + q\psi^\delta - f|^2 \}$$

over $\psi^\delta \in S_\delta$, $\underline{v}^h \in \mathcal{V}_{\sim h}$ where r is the distance to the vertex (Fig. 1). Appropriate choices for $\alpha > 0$ will be discussed in the next section.

An equivalent statement of this variational principle involves the bilinear form

$$(2.5) \quad B_\alpha((\psi,\underline{v}),(\xi,\underline{w})) = \int_\Omega \{ (\text{grad } \psi - \underline{v}) \cdot (\text{grad } \xi - \underline{w}) + r^\alpha (\text{div } \underline{v} + q\psi)(\text{div } \underline{w} + q\xi) \}$$

and the functional

$$(2.6) \qquad F_\alpha(\xi,\underline{w}) = \int_\Omega r^\alpha f (\text{div } \underline{w} + q\xi) \, .$$

In particular, the minimum of (2.4) is characterized by the following variational principle. Find functions (2.3) such that

$$(2.7) \qquad B_\alpha((\phi_\delta,\underline{u}_h),(\psi^\delta,\underline{v}^h)) = F_\alpha(\psi^\delta,\underline{v}^h)$$

holds for all $\psi^\delta \in S_\delta$ and $\underline{v}^h \in \mathcal{V}_h$.

Observe that $B_\alpha(\cdot,\cdot)$ and $F_\alpha(\cdot)$ are continuous on $I = \overset{01}{H}{}^1(\Omega) \times \underline{W}^1_{-\alpha}(\Omega)$. Indeed, taking

$$(2.8) \qquad |||(\phi,\underline{u})||| = \{ \|\text{grad } \phi\|^2_{[L^2(\Omega)]^2} + \|\phi\|^2_{L^2[\Omega]} + \|\underline{u}\|^2_{1,\alpha} \}^{\frac{1}{2}},$$

as the norm on I, then as an easy consequence of the Schwarz inequality we obtain with $\alpha \geq 0$ and $q \in L^\infty(\Omega)$

$$(2.9) \qquad |B_\alpha((\phi,\underline{u}),(\psi,\underline{v}))| \leq C \, |||(\phi,\underline{u})||| \; |||(\psi,v)|||$$

and for $f \in L^2(\Omega)$ in (2.6).

$$(2.10) \qquad |F_\alpha(\psi,\underline{v})| \leq C \;|||\,(\psi,\underline{v})\,|||\,.$$

Thus (2.7) has a meaning, and in fact, is equivalent to a Hermitian nonnegative definite system of algebraic equations once a basis is chosen for $S_\delta \times \underline{V}_h$. If we assume that (1.1) - (1.2) has a unique solution for each $f \in L^2(\Omega)$, then the algebraic system is positive definite. This will be the case, for example, if q is never equal to an eigenvalue of the Laplacian with Dirichlet boundary conditions.

3. Analysis of Errors

Because of the singularities in the solution to (1.1) - (1.2), piecewise linear functions are perhaps the most practical choices for S_δ and $\underline{V}_h$, and in this section we shall restrict attention to this case. The grids for the space S_δ of scalars need not coincide with the grids for the space $\underline{V}_h$ of vectors, and as we shall see subsequently, there will be important reasons for this.

We recall that approximation theory asserts that there is a positive number C satisfying the following [5].

Given any $\underline{u} \in \underline{W}^k_\alpha$ and $\phi \in H^\ell(\Omega) \cap \overset{O1}{H}(\Omega)$ there are $\hat{\underline{u}}_h \in \underline{U}_h$ and $\hat{\phi}_\delta \in S_\delta$ such that

$$(3.1) \qquad \|\,\underline{u} - \hat{\underline{u}}_h\,\|_{t,\alpha} \leq Ch^\mu \|\,\underline{u}\,\|_{k,\alpha} \qquad \mu = \min\,\{k-t, \tfrac{\pi}{\theta_0} - 1 + \tfrac{\alpha}{2}\,\}$$

$$(3.2) \qquad \|\,\phi - \hat{\phi}_\delta\,\|_{t,0} \leq C\delta^{\ell-t} \|\,\phi\,\|_{\ell,0} \qquad (1)$$

for $0 \leq t \leq k \leq 2$, $t \leq \ell$ and $0 < h < \underline{h}_0$, $0 < \delta \leq \delta_0$. The goal of this section is to develop similar estimates for the errors

(3.3)
$$\underline{e} = \underline{u} - \underline{u}_h, \qquad \varepsilon = \phi - \phi_\delta$$

in the least squares approximation.

Crucial to our error analysis is the regularity of the solutions to (1.1) - (1.2) in appropriate Sobolev spaces. For simplicity we shall assume that the interior angle is re-entrant, i.e., $\pi < \theta_0 < 2\pi$. Due to [5] for given $f \in L^2(\Omega)$ and q not being an eigenvalue there exists exactly one solution ϕ of (1.1) - (1.2) in $\overset{o}{H}{}^1(\Omega) \cap W^1_{-\beta}(\Omega)$, $\beta = \frac{2\pi}{\theta_0}$. Therefore due to Kondratiev [6] the following important regularity result holds.

Theorem 1: Let q be not an eigenvalue of (1.1), (1.2). Then there are

(3.4)
$$\alpha > 2t + 2 - 2s_0, \qquad s_0 = \frac{\pi}{\theta_0}, \qquad C_R > 0$$

such that for any $f \in W^t_\alpha(\Omega) (t = 0, 1)$ the solution ϕ of (1.1) - (1.2) satisfies

(3.5)
$$\| \phi \|_{t+2,\alpha} \leq C_R \| f \|_{t,\alpha} \leq C_R \| f \|_{t,0}$$

Moreover, for $1 < s < 1 + s_0$

(3.6)
$$\| \phi \|_{s,0} \leq C_R \| f \|_{0,0} .$$

It is an easy consequence of (2.7) that $\{\underline{u}_h, \phi_\delta\}$ is a best approximation to $\{\text{grad } \phi, \phi\}$ in the norm generated by $B_\alpha(\cdot, \cdot)$. Thus we have the following consequence of Theorem 1 and the approximation properties (3.1) - (3.2) together with (2.9).

---

[1] Here $\| \cdot \|_{k,\alpha}$ denotes the norm associated with $W^k_{-\alpha}$. In addition, $\| \cdot \|_{t,0}$ denotes the (unweighted) norm on $H^t(\Omega)$.

Lemma 1. There is a constant $C > 0$ depending only on C_R and

$$(3.7) \qquad \alpha \geq 4 - \frac{2\pi}{\theta_0}$$

such that

$$(3.8) \qquad B_\alpha((\varepsilon, \underline{e}), (\varepsilon, \underline{e}))^{\frac{1}{2}} \equiv |||(\varepsilon, \underline{e})|||_\alpha \leq C(h + \delta^{s_0}) \|f\|_{1,0}.$$

Our next estimate is in the weighted dual norm defined as follows

$$(3.9) \qquad \|\psi\|_\alpha^* = \sup_\eta \left\{ \frac{\int_\Omega r^\alpha \psi \eta}{|||\eta|||_{1,\alpha+4}} \right\}.$$

Lemma 2. There is a constant $C > 0$ depending only on $\alpha \neq 1$ which satisfies (3.4), (3.8) and $C_R > 0$ such that

$$(3.10) \qquad \|\text{div } \underline{e} + q\varepsilon\|_\alpha^* \leq C(\delta^{s_0} + h) |||(\varepsilon, \underline{e})|||_\alpha.$$

Proof. Let $\eta \in W_{\alpha+4}^1$ be given, and consider the solution ξ to

$$(3.11) \qquad \Delta\xi + q\xi = \eta \text{ in } \Omega, \ \xi = 0 \text{ on } \Gamma.$$

Letting

$$\underline{p} = \text{grad } \xi$$

we have from (3.5) with α satisfying (3.4) and (3.7)

$$(3.12) \qquad \|\underline{p}\|_{2,\alpha+4} \leq \|\xi\|_{3,\alpha+4} \leq C_R \|\eta\|_{1,\alpha+4}.$$

Also

$$B_\alpha((\varepsilon,\underline{e}),(\xi,\underline{p})) = \int_\Omega r^\alpha \eta(\operatorname{div}\underline{e}+q\varepsilon).$$

Using orthogonality this becomes

(3.13) $$B_\alpha((\varepsilon,\underline{e}),(\xi-\hat{\xi}_\delta,\underline{p}-\hat{\underline{p}}_h)) = \int_\Omega r^\alpha \eta(\operatorname{div}\underline{e}+q\varepsilon)$$

Thus

(3.14) $$|||(\varepsilon,\underline{e})|||_\alpha \; |||(\xi-\hat{\xi}_\delta,\underline{p}-\hat{\underline{p}}_h)|||_\alpha \geq \left| \int r^\alpha \eta(\operatorname{div}\underline{e}+q\varepsilon) \right|.$$

Using the approximation properties (3.1) – (3.2) and taking the sup over η with $\|\eta\|_{1,\alpha+4} \leq 1$ we obtain (3.10) from (3.14) since the solution ξ of (3.11) satisfies

$$\|\xi\|_{H^{1+s_0}(\Omega)} \leq c\|\eta\|_{L^2(\Omega)} \leq \tilde{c}\|\eta\|_{W_{\alpha+4}^1}.$$

The latter inequality holds due to the continuity of the imbedding of W_α^1 into $L^2 (\alpha \neq 1)$ ([7], p.287). The cone property for Ω is needed for this result.

<u>Lemma 3.</u> There is a constant $C > 0$ depending only on $\alpha \geq 4 - \dfrac{2\pi}{\theta_0}$, $\|q\|_{L_\infty(\Omega)}$ and C_R such that

(3.15) $$\|\varepsilon\|_{L^2(\Omega)} \leq C(\|\operatorname{div}\underline{e}+q\varepsilon\|_\alpha^* + \delta^{s_0}|||(\varepsilon,\underline{e})|||_\alpha)$$

Proof. The first step is to solve for ϵ in $L^2(\Omega)$

$$(3.16) \qquad \Delta\eta + q\eta = \epsilon \quad \text{in } \Omega, \ \eta = 0 \text{ on } \Gamma.$$

For this system we have from (3.4) and (3.5)

$$(3.17) \qquad \|\eta\|_{2,\sigma} \leqq c\|\epsilon\|_{0,\sigma} \leqq c\|\epsilon\|_0$$

provided $2 - \dfrac{2\pi}{\theta_0} < \sigma$ for $\pi \leqq \theta_0 \neq 2\pi$. Also it follows from (3.16) that

$$\|\epsilon\|_0^2 = \int_\Omega \epsilon \, (\Delta\eta + q\eta) \, ,$$

which after integrating by parts becomes

$$(3.18) \qquad \|\epsilon\|_0^2 = \int_\Omega \{-\text{grad } \epsilon \cdot \text{ grad } \eta + q\epsilon\eta\}$$

But

$$(3.19) \qquad B_\alpha((\epsilon,\underline{e}),(\eta,\underline{0})) = \int_\Omega (\text{grad } \epsilon - \underline{e}) \cdot \text{grad } \eta + r^\alpha(\text{div } \underline{e} + q\epsilon)q\eta\}.$$

Thus combining (3.18) – (3.19) and using orthogonality we obtain

$$\|\epsilon\|_0^2 = \int \left[-\underline{e}\cdot\text{grad } \eta + q\epsilon\eta\right] + \int_\Omega r^\alpha \, (\text{div } \underline{e} + q\epsilon) \, q\eta - B_\alpha((\epsilon,\underline{e}),(\eta-\hat{\eta}_\delta,0)).$$

Integrating the first term on the right by parts gives the simpler form

$$(3.20) \qquad \|\epsilon\|_0^2 = \int_\Omega \left[(r^\alpha q + 1)(\text{div } \underline{e} + q\epsilon)\eta\right] - B_\alpha((\epsilon,\underline{e}),(\eta-\hat{\eta}_\delta)0).$$

The second term on the right hand side of (3.20) gives the second term in (3.15). Thus our task is to estimate the first term. To do this we use the Hardy-Littlewood inequality [7, p. 286]. This inequality states with $D \equiv \mathrm{grad}$ that

$$(3.21) \qquad C \| r^{\sigma/2} D^2 \eta \|_{0,0} \geq \| r^{\frac{\sigma-2}{2}} D\eta \|_{0,0}$$

provided $2 - \frac{2\pi}{\theta_0} < \sigma$. Note that

$$\left| \int_\Omega (r^\alpha q + 1)(\mathrm{div}\,\underline{e} + q\varepsilon)\eta \right| \leq C \int_\Omega |\mathrm{div}\,\underline{e} + q\varepsilon|\,|\eta|$$

$$(3.22) \qquad = C \int_\Omega r^\alpha |\mathrm{div}\,\underline{e} + q\varepsilon|\,|r^{-\alpha}\eta|.$$

Thus using (3.9) we see that the right-hand side of (3.22) is bounded above by

$$\| \mathrm{div}\,\underline{e} + q\varepsilon \|_\alpha^* \, \| r^{\frac{\alpha+4}{2}} \mathrm{grad}(r^{-\alpha}\eta) \|_{0,0}$$

which in turn is bounded above by

$$\| \mathrm{div}\,\underline{e} + q\varepsilon \|_\alpha^* \, \| r^{-\frac{\alpha-4}{2}} \mathrm{grad}\,\eta \|_{0,0}.$$

To use (3.17) we must bound the second term in the above by $\|\eta\|_{2,\sigma}$, and to do this we take

$$(3.23) \qquad 4 - \alpha = \sigma - 2$$

in the Hardy-Littlewood inequality. This gives

$$(3.24) \qquad \left| \int_\Omega (r^\alpha q + 1)(\operatorname{div} \underline{e} + q\varepsilon)\eta \right| \leq C \| \operatorname{div} \underline{e} + q\varepsilon \|_\alpha^* \| r^{\sigma/2} D^2 \eta \|_{0,0} .$$

But α satisfies (3.7) and thus by (3.23) and (3.17)

$$(3.25) \qquad \| r^{\sigma/2} D^2 \eta \|_0 \leq C \| \varepsilon \|_0$$

provided

$$(3.26) \qquad 2\frac{2\pi}{\theta_0} < \sigma \leq 2 + \frac{2\pi}{\theta_0} .$$

Combining (3.24) – (3.25) we obtain (3.15) from (3.20). Note that the approximation property (3.1) for $k = 2$, $t = 1$ is only quasi-optimal if α satisfies (3.8).

Inserting (3.8) and (3.10) into (3.15) we obtain an L^2-estimate for ε .

Theorem 2: There is a constant C depending only on $\|q\|_{L_\infty}$, $\alpha \geq 4 - 2\pi/\theta_0$ and C_R such that

$$(3.27) \qquad \| \phi - \phi_\delta \|_0 \leq C(\delta^{s_0} + h)^2 \|f\|_1 .$$

Remark. For optimal accuracy we take

$$(3.28) \qquad \delta = h^{1/s_0} ;$$

since $s_0 = \pi/\theta_0$, $\pi < \theta_0 < 2\pi$, the grid for the scalar field ϕ must be finer than that for the vector field $\underline{u}$.

We now use Lemma 1 to estimate $\underline{e} = \underline{u} - \underline{u}_h$. To do this we shall need for S_δ to have an inverse property. More precisely, we shall assume there is a number $0 < C < \infty$ independent of δ such that

$$(3.29) \qquad \|\psi_\delta\|_1 \leq C\delta^{-1} \|\psi_\delta\|_0 \quad \text{all } \psi_\delta \in S_\delta.$$

Theorem 3: Let (3.28) and (3.29) hold. Then for $\alpha \geq 4 - \dfrac{2\pi}{\theta_0}$

$$(3.30) \qquad \|\underline{u} - \underline{u}_h\|_0 \leq Ch\| f \|_1$$

Proof. Let $\psi_\delta \in S_\delta$ satisfy

$$(3.31) \qquad \|\phi - \psi_\delta\|_r \leq C\delta^{1+s_0-r} \|\phi\|_{1+s_0}$$

for $r = 0$ and $r = 1$. Then from

$$\|\underline{e}\|_0^2 - \|\operatorname{grad} \varepsilon\|_0^2 \leq \|\underline{e} - \operatorname{grad} \varepsilon\|_0^2$$

we obtain

$$(3.32) \qquad \|\underline{e}\|_0 \leq C(\|\operatorname{grad} \varepsilon\|_0 + \||(\varepsilon,\underline{e})\||_\alpha).$$

But

$$(3.33) \qquad \|\operatorname{grad} \varepsilon\|_0 \leq \|\operatorname{grad}(\phi - \psi_\delta)\|_0 + \|\operatorname{grad}(\psi_\delta - \phi_\delta)\|_0.$$

We use (3.31) to estimate the first term, and apply the inverse inequality (3.29) to the second term to get

$$(3.34) \qquad \|\operatorname{grad} \varepsilon\|_0 \leq C\delta^{s_0}\|\phi\|_{1+s_0} + C\delta^{-1}\|\psi_\delta - \phi_\delta\|_0.$$

But

(3.35) $\|\psi_\delta - \phi_\delta\|_0 \leq \|\phi - \phi_\delta\|_0 + \|\phi - \psi_\delta\|_0 \leq c\delta^{1+s_0}\|f\|_0 .$

Combining these estimates we obtain (3.30) with (3.28).

Remark (i). To get the full rate of convergence it is necessary to refine the grid for the scalar potential ϕ, taking

$$\delta = h^{1/s_0} .$$

This gives

$$\|\varepsilon\|_0 = O(h^2), \quad \|\underline{e}\|_0 = O(h).$$

(ii). For mixed boundary conditions

(3.36) $\phi = 0 \quad \text{on} \quad \Gamma_D, \quad \dfrac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_N$

where ν denotes the outer normal at the boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. The solution ϕ of (1.1) behaves like $r^{\pi/2\theta_0}\Phi(\phi)$ at the singular points, where Φ is analytic. Therefore the above analysis holds as well for the mixed boundary value problem (1.1), (3.36) by choosing the weight

$$\alpha \geq 4 - \frac{\pi}{\theta_0}$$

in the least squares scheme (2.4) and refining the mesh as $\delta = h^{2\theta_0/\pi}$.

(iii). Since the solutions of crack problems behave like the solutions of mixed boundary value problems with smooth boundary, our weighted least squares method can also be applied to crack problems. Choosing $\alpha \geq 3$, $\delta = h^2$ we obtain with piecewise linear test and trial functions

(3.37) $\|\varepsilon\|_0 = O(h^2); \quad \|\underline{e}\|_0 = O(h).$

The standard Galerkin procedure gives for ε the same error estimate if $\delta = h^2$. But in order to obtain (3.37) for $\underline{e}$ with the Galerkin procedure, one has to use special singularity functions as test and trial functions.

(iv). The results in this paper do not apply to the general three-dimensional case because the regularity results used are not known for three-dimensional domains with arbitrary corners.

REFERENCES

[1] G. J. FIX, M. D. GUNZBURGER, and R. A. NICOLAIDES, "On finite element methods of the least squares type," *Comput. Math. Appl.*, v. 5, 1979, pp. 87-98.

[2] G. J. FIX, and M. D. GUNZBURGER, "On numerical methods for acoustic problems," *Comput. Math. Appl.*, v. 6, 1980, pp. 265-278.

[3] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[4] Y. Lee, "Numerical approximations with least squares finite element methods for selected problems in fracture mechanics," Ph.D. Thesis, Department of Mechanical Engineering, Carnegie-Mellon University.

[5] I. BABUSKA, and M. B. ROSENWEIG, "A finite element scheme for domains with corners," *Numer. Math.*,v. 20, 1972, pp. 1-21.

[6] V. A. KONDRATIEV, "Boundary problems for elliptic equations in domains with conical or angular points," *Trans. Moscow Math. Soc.*, v. 16, 1967, pp. 227-313.

[7] J. NEČAS, Les methodes directes en théorie des equations elliptiques," Paris: Masson, 1967.